# MATH 512 HOMEWORK 5

Due Friday, May 3.

Problems 1-3 go over the following theorem of Laver:

**Theorem 0.1.** *Suppose that in $V$, $\kappa$ is supercompact. Then there is a generic extension, in which $\kappa$ remains supercompact and for any $\kappa$-directed closed forcing $\mathbb{Q}$, forcing with $\mathbb{Q}$ preserves the supercompactness of $\kappa$. Such a cardinal is said to be indestructibly supercompact.*

Suppose $V \models \kappa$ is supercompact. Let $f : \kappa \to V_\kappa$ be a Laver function. Define $\mathbb{P} = \langle \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa \rangle$ to be an iteration of length $\kappa$ with Easton support (i.e. direct limits at regular $\alpha$'s and inverse limits at singular $\alpha$'s.) such that for each $\alpha$, we define $\dot{\mathbb{Q}}_\alpha$ as follows:

1. $f(\alpha) = (\lambda, \dot{\mathbb{Q}})$, where $\lambda \in Ord$, $\dot{\mathbb{Q}}$ is $\mathbb{P}_\alpha$-name for an $\alpha$-directed closed poset, and for all $\beta < \alpha$, if $f(\beta) = (a_0, a_1)$ where $a_0 \in Ord$, then $a_0 < \alpha$. In this case set $\dot{\mathbb{Q}}_\alpha$.
2. Otherwise, set $\dot{\mathbb{Q}}_\alpha$ to be a name for the trivial poset.

Then $\mathbb{P}$ is $\kappa$-c.c. with cardinality $\kappa$. So, it has no effect on cardinals and cofinalities above $\kappa$. Let $G$ be $\mathbb{P}$-generic over $V$. Suppose that in $V[G]$, $\mathbb{Q}$ is a $\kappa$-directed closed notion of forcing and $\lambda \geq \kappa$ be such that $\dot{\mathbb{Q}} \in H_\lambda$, where $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name in $V$ for the above poset. The problems below give the argument that after forcing with $\mathbb{Q}$, $\kappa$ is $\lambda$-supercompact.

**Problem 1.** *Let $H$ be $\mathbb{Q}$-generic over $V[G]$ and let $\mu = 2^{2^\lambda}$. Show that there is an elementary embedding $j : V[G] \to M[G*H*K]$, for some generic filter $K$, such that $j$ extends a $\mu$-supercompact embedding $j_0 : V \to M$.*

**Problem 2.** *Show that we can lift $j$ from the above problem to an embedding $j^* : V[G*H] \to M[G*H*K*K']$ for some generic filer $K'$. Here $j^*$ will be defined in $V[G*H*K*K']$.*

**Problem 3.** *Use $j^*$ from the above problem to define a normal measure $U$ on $\mathbb{P}_\kappa(\lambda)$, such that $U \in V[G]$. Conclude that in $V[G]$, $\kappa$ is $\lambda$-supercompact.*

**Problem 4.**

1. *Show that every c.c.c forcing has the $\omega_1$-covering property.*
2. *Let $\gamma \geq \omega_1$. Show that $Add(\omega, 1) * \dot{Col}(\omega_1, \gamma)$ has the $\omega_1$-covering property.*

**Problem 5.** *Suppose $\mathbb{P}$ has the $\omega_1$-covering property and the $\omega_1$-approximation property. Suppose also $\lambda \geq \omega_1$ and $1_{\mathbb{P}} \Vdash \dot{f} : \omega_1 \to \mathcal{P}_{\omega_1}(\lambda)$ is continuous and cofinal. Let $M \in \mathcal{P}_{\omega_2}(H_\theta)$ be a substructure with $\dot{f}, \lambda, \mathbb{P} \in M$, and $G \subset \mathbb{P}$ be $M$-generic and $f = \dot{f}_G$.*

(1) *Show that* $f : \omega_1 \to \mathcal{P}_{\omega_1}(M \cap \lambda)$ *is continuous and cofinal.*
(2) *Show that for every* $x \subset M, |x| < \omega_1$, *there is* $y \in M$ *such that* $|y| < \omega_1$ *and* $x \subset y$.